



AP* Calculus Review

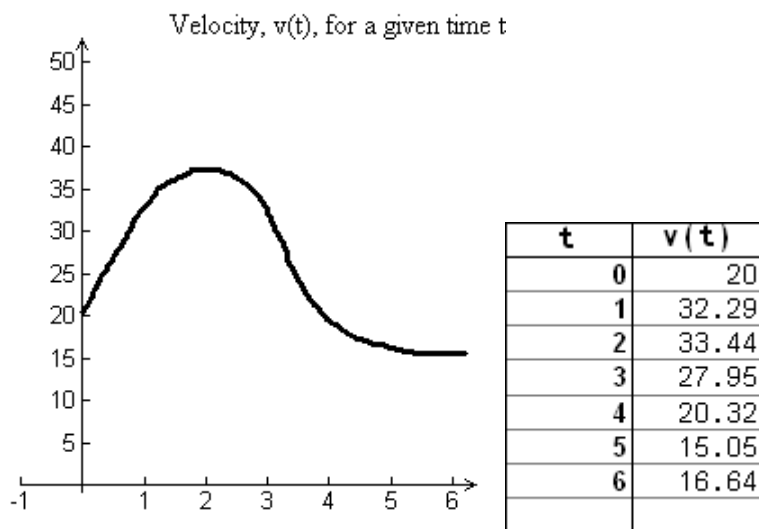
Approximating Definite Integrals

Teacher Packet



Approximating Definite Integrals (Riemann and Trapezoid Sums)

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The graph above represents the velocity of an object in motion, in ft/sec, for any time t , in seconds.

To approximate distance traveled in 6 seconds, a Riemann sum may be used. Six subdivisions would make maximum use of the data in the table. However, one could also use 2 or 3 subdivisions. When subdivisions of equal width are to be used, calculate the width of the subdivision (Δt) using the formula $\frac{\text{ending value} - \text{starting value}}{\# \text{ of subdiv}}$. The

process of the Riemann sums involves adding together the products of Δt 's and velocities. You are expected to use left side, right side, or midpoint values of the velocity, as required by the problem.

A right side Riemann sum with three subdivisions would look like:

$$\text{Distance} = 2(33.44 + 20.32 + 16.64) = 140.8 \text{ feet}$$

A left side Riemann sum with three subdivisions would look like:

$$\text{Distance} = 2(20 + 33.44 + 20.32) = 147.52 \text{ feet}$$

A midpoint Riemann sum with three subdivisions would look like:

$$\text{Distance} = 2(32.29 + 27.95 + 15.05) = 150.58 \text{ feet}$$

Note for midpoints: To find the velocity value, first find the midpoint of the time interval and then use the corresponding velocity. The midpoint value is not found by finding the midpoint of the velocity values. The midpoint method may not be applicable to all data tables. On this table the midpoint method would not work with 6 subdivisions or 2 subdivisions.



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Another method of approximating this distance is to use trapezoids. The formula for that area of a trapezoid is $\frac{h(b_1 + b_2)}{2}$. Subdivide the region from 0 to 6 and apply the area

formula for a trapezoid in each region.

Suppose you use 3 equal subdivisions, the approximation would be:

$$\frac{2(20 + 33.44)}{2} + \frac{2(33.44 + 20.32)}{2} + \frac{2(20.32 + 16.64)}{2} = 144.16 \text{ feet}$$

NOTE: These examples have all assumed subdivisions of equal width. In many data problems the subdivisions cannot be equal in width. In those problems, the areas will have to be calculated separately and then added together.

Example:

t (seconds)	0	8	11	17
v (feet/second)	34	65	67	43

Using three subdivisions you can use left or right Riemann sums (or a trapezoid sum).

Left = $8(34) + 3(65) + 6(67) = 869$ feet

Right = $8(65) + 3(67) + 6(43) = 979$ feet

Relative size of the approximations

If your function is increasing on the entire interval,
left sum < true value < right sum.

If your function is decreasing on the entire interval,
right sum < true value < left sum.

If your function is concave down on the entire interval,
trapezoid sum < true value < midpoint sum.

If your function is concave up on the entire interval,
midpoint sum < true value < trapezoid sum.

While this information is relating to a problem of finding distance, you can use all of these methods to approximate any definite integral.



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Sample Problems

Multiple Choice – No Calculator

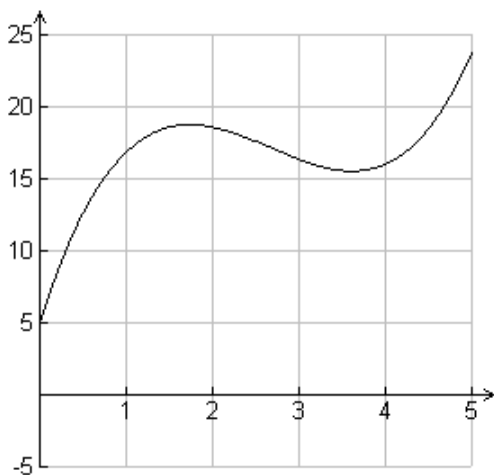
1. If $\int_1^{11} \sqrt{x} \, dx$ is approximated with 5 midpoint rectangles of equal width, then the approximation is

(A) $\sqrt{2} + 2 + \sqrt{6} + 2\sqrt{2} + \sqrt{10}$ (B) $2(\sqrt{2} + 2 + \sqrt{6} + 2\sqrt{2} + \sqrt{10})$

(C) $2(1 + \sqrt{3} + \sqrt{5} + \sqrt{7} + 3)$ (D) $2(\sqrt{3} + \sqrt{5} + \sqrt{7} + 3 + \sqrt{11})$

(E) $2\left(\frac{1 + \sqrt{3}}{2} + \frac{\sqrt{3} + \sqrt{5}}{2} + \frac{\sqrt{5} + \sqrt{7}}{2} + \frac{\sqrt{7} + 3}{2} + \frac{3 + \sqrt{11}}{2}\right)$

2.



The graph of f is given. An estimate of $\int_0^5 f(x) \, dx$ is

- (A) 16 (B) 19 (C) 80 (D) 95 (E) 100



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3. Approximate $\int_2^{14} 2x \, dx$ using a left Riemann sum and 4 equal subdivisions.

- (A) 52 (B) 76 (C) 156 (D) 192 (E) 228

4.

Time (milliseconds)	0	5	10	15	20	25	30
Light output (millions of lumens)	0	.2	.6	2.6	4.2	3	1.8

The data in the table represents the rate of light output of a flash bulb at a given time. Using a midpoint approximation with 3 equal subdivisions estimate the total light output of the bulb, measured in million lumen-milliseconds.

- (A) 5.8 (B) 48 (C) 57 (D) 58 (E) 66

5. The table below gives data points for a continuous function f on $[2, 10]$.

x	2	5	7	8	10
$f(x)$	3	1	5	3	5

Using subdivisions $[2, 5]$, $[5, 8]$ and $[8, 10]$, what is the trapezoid approximation of $\int_2^{10} f(x) \, dx$?

- (A) 18 (B) 20 (C) 22 (D) 24 (E) 40



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6. The table below gives data points for a continuous function f on $[3, 12]$.

x	3	5	8	12
$f(x)$	2	4	6	4

What is the absolute value of the difference between a right sum and a left sum when approximating $\int_3^{12} f(x) dx$ with 3 intervals?

- (A) 0 (B) 2 (C) 4 (D) 6 (E) 7
7. Estimate $\int_{\frac{\pi}{2}}^{\pi} \sin(x) dx$ using a right Riemann sum with 2 equal subdivisions.

- (A) $\pi(1+0)$ (B) $\frac{\pi}{4}\left(-\frac{\sqrt{2}}{2}-1\right)$ (C) $\frac{\pi}{4}\left(\frac{\sqrt{2}}{2}+1\right)$
- (D) $\frac{\pi}{4}\left(\frac{\sqrt{2}}{2}-1\right)$ (E) $\frac{\pi}{4}\left(\frac{\sqrt{2}}{2}+0\right)$



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Free Response 1 – No Calculator

An object moves in a straight line during $0 \leq t \leq 12$ seconds. The velocity (m/sec) and acceleration (m/sec²) are differentiable functions. The table shows selected values of the functions.

t (sec)	0	2	5	6	10	12
v (m/sec)	0	4	8	6	10	30
a (m/sec ²)	1	3	2	4	1	3

- (a) Using a trapezoid sum and 5 intervals estimate $\int_0^{12} v(t) dt$.
- (b) Using correct units explain the meaning of $\frac{1}{12} \int_0^{12} v(t) dt$.
- (c) Using a left sum and 5 intervals approximate $\int_0^{12} a(t) dt$. Include units in your answer. What is the true value of $\int_0^{12} a(t) dt$?
- (d) Explain why acceleration must equal $\frac{1}{2}$ for some value on $[2, 6]$.



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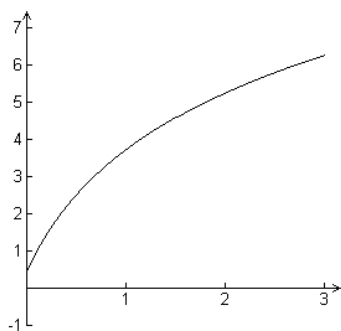
Multiple Choice – Calculator Allowed

1. A radar gun was used to record the velocity (m/sec) of a runner. Data points for that event are recorded in the table. Use a trapezoid sum to approximate distance covered in 8 seconds.

t (sec)	0	4.3	6.5	8
v (m/sec)	0	10.83	12.46	9.32

- (A) 42.516 (B) 65.239 (C) 87.961
- (D) 130.477 (E) 260.954
2. What is the absolute value of the difference between the exact value for $\int_{-1}^7 (3x-1) dx$ and a midpoint approximation with 4 equal subdivisions?
- (A) 0 (B) 22 (C) 40 (D) 64 (E) 88

3.



The graph represents a continuous function f on $[0, 3]$. If L_3 , R_3 , T_3 and M_3 represent the approximations of $\int_0^3 f(x) dx$ - left sum, right sum, trapezoid sum, and midpoint sum, respectively. Which of the following is not true?

- (A) $L_3 < R_3$ (B) $T_3 < R_3$ (C) $T_3 < M_3$
- (D) $M_3 < L_3$ (E) $M_3 < R_3$



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Free Response – Calculator Allowed

1. Let $R(t)$ represent the rate of change of the value of a stock, in dollars per day over a 12 day period.

t (days)	0	2	5	9	11	12
$R(t)$ (\$/day)	10	22	15.77	13.29	12.70	12.48

- (a) Using a right Riemann sum and 5 intervals estimate $\int_0^{12} R(t) dt$. Explain the meaning of this integral using correct units.
- (b) What is the average rate of change in $R(t)$ over $[0, 12]$?
- (c) Using your work from part (a), what is the average value of $R(t)$ over $[0, 12]$? Include units.
- (d) Let the function defined by $Q(t) = \frac{30t}{1+t^2} + 10$ be a model for the rate of change in the value of the stock on the interval $[0, 12]$. For what value(s) of t does the instantaneous rate of change in Q equal the average rate change of Q ? Show the equation that leads to the answer.



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Free Response – Calculator Allowed

2. Let $F(t) = \frac{200t+10}{2t^2+15}$ be a function that models the rate, in gallons per hour, at which a water tank is being filled over a 24-hour period.

Let $E(t) = \frac{20t^2 - 50t + 160}{2t^2 + 15}$ be a function that models the rate, in gallons per hour, at which a water tank is being emptied over the same 24-hour period. Assume the tank contained 50 gallons at $t = 0$.

- (a) Set-up the approximation for $\int_0^{24} F(t) dt$ using midpoint sums and 4 equal subdivisions.

- (b) Find $F(1) - E(1)$. Explain what the answer tells you about the water in the tank. Include correct units.

- (c) Evaluate $\int_0^1 (F(t) - E(t)) dt$. Explain what the answer tells you about the water in the tank. Include correct units.

- (d) On what interval(s) is the level of water in the tank rising? Explain your answer.